

# ANOTHER DIMINIMAL MAP ON THE TORUS

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ABSTRACT. This note adds one diminimal map on the torus to the published set of 55. It also raises to 15 the number of vertices for which all diminimal maps on the torus are known.

## 1. INTRODUCTION

Riskin [4][5] found two diminimal maps on the torus (denoted R1 and R2 in this note). Henry [2][3], using a computer found 53 more (H1 through H53). All diminimal maps on the torus with fewer than 10 vertices or fewer than 10 faces were found using this technique. But Henry was not able to complete the generation process due to time constraints.

Using a completely different computer technique we generated all polyhedral maps on the torus with 15 or fewer vertices and checked each to see if it was diminimal. While this technique found all diminimal maps on the torus with 15 or fewer vertices, there may be other diminimal maps on the torus with more than 15 vertices.

## 2. DEFINITIONS

Following [3] we use these definitions. We consider only graphs with no multiple edges and no loops. If  $G$  is a graph embedded in a surface, then the closure of each connected component of the complement of the graph is called a *face*. An embedded graph together with its embedding is called a *map* if each vertex of the graph has degree at least 3 and each face is a closed 2-cell.

If the intersection of two distinct faces of a map is empty, a vertex, or an edge then the faces are said to *meet properly*. If each face is simply connected and each pair of faces meet properly, then the map is a *polyhedral map*. We refer to a polyhedral map on the torus as a *toroidal polyhedral map* or TPM.

The operation *edge removing* is the process of obtaining one map from another by removing a single edge. If the removal creates a vertex of degree 2 then the two edges adjacent to that vertex are coalesced into one edge in the new map. Let  $G$  be a TPM and let  $G'$  be the map obtained from  $G$  by removing edge  $e$  from  $G$ . If  $G'$  is also a TPM, then edge  $e$  is called *removable*.

The operation *edge shrinking* is the process of obtaining one map from another by contracting an edge so that edge's two vertices become one vertex. If the contraction creates a two-sided face in the new map then the two edges of the two-sided face are replaced with a single edge. Let  $G$  be a TPM and let  $G'$  be the map obtained from  $G$  by shrinking edge  $e$  in  $G$ . If  $G'$  is also a TPM, then edge  $e$  is called *shrinkable*.

If  $G$  is a TPM with no shrinkable edges and no removable edges then  $G$  is called *diminimal*.

### 3. GENERATING TPMS

The author has developed the program *surftri* [6] for generating embeddings on surfaces. This program is based on the program *plantri* [1] written by Brinkmann and McKay which generates planar graphs. It is easy to restrict *surftri* to generate only *polyhedral maps*. As a test of *surftri* we checked that all the known diminimal TPMs were among the TPMs with 15 or fewer vertices. The number of TPMs checked is shown in Table 1. About three CPU years were required to generate the TPMs with up to 15 vertices. In the process we discovered an additional diminimal TPM (S1). All the known diminimal TPMs are shown in Figs. 2, 3, and 4.

A list of the 56 known diminimal TPMs (H1, . . . , H53, R1, R2, S1) is in Figure 1. Each line of the list represents one diminimal TPM. The format used is taken from *surftri* [6] which generalizes the format in *plantri* [1]. The vertices are labeled by single letters. The number is the number of vertices for the embedding. For each vertex there is a list of its neighbors in cyclic order. So

7 bcdefg, agdfec, abegfd, acfbge, adgcbf, aebdcg, afcedb

represents  $K_7$  embedded on the torus. This embedding has 7 vertices. The vertex **a** has 6 neighbors **bcdefg** in cyclic order. The vertex **b** also has 6 neighbors **agdfec** in cyclic order. The final vertex **g** has neighbors **afcedb**.

Vertices	TPMs	Diminimal TPMs
7	1	1
8	33	2
9	4713	11
10	442429	19
11	28635972	15
12	1417423218	5
13	58321972887	2
14	2102831216406	1
15	68781200467456	0

TABLE 1. Counts of TPMs and diminimal TPMs by number of vertices

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7 bcdefg,agdfec,abegfd,acfbge,adgcbf,aebdcg,afcedb
8 bcde,aefgdh,ahgf,afhbg,aghfb,behdcg,bfced,bdfec
8 bcd,aefgh,ahegf,afhge,bdgchf,behdcg,bfcedh,bgdfec
9 bcde,afghi,aifh,ahge,adihf,becg,bfid,bdcei,bhegc
9 bcd,aefgh,ahei,aihge,bdgc,behi,biedh,bgdfc,cgfd
9 bcd,aefgh,ahig,afhe,bdhgi,bidg,bfcedh,bgedic,chfe
9 bcd,aefgh,aheg,agfi,bicf,behdg,bfdcih,bgfc,dhge
9 bcd,aefgh,aig,afhe,bdhgif,beidg,bfcedh,bgedi,chfe
9 bcde,afghi,aifd,achg,agihf,behcg,bfied,bdfei,bhegc
9 bcd,aefgh,ahegf,afhge,bdgc,biidg,bfcedh,bgdic,ehf
9 bcd,aefgh,ahig,agfhe,bdhgi,biidg,bfcedh,bgedfc,cfe
9 bcd,aefgh,ahfi,agfhe,bdhif,behdg,bfdih,bgedfc,ceg
10 bcde,afgh,ahfi,aihge,agif,becj,bjed,bdjc,cejd,fhig
10 bcd,aefgh,ahij,ajhe,bdgi,biij,bjeh,bgdfc,cfe,cgfd
10 bcd,aefg,ahei,aihj,bjcf,behi,bijh,cgdf,cjgfd,dgie
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10 bcd,aefgh,aig,afje,bdjif,behdg,bfcj,bjfi,che,dhge
10 bcd,aefg,ahi,ajge,bdgi,hj,bji,biedh,cgje,cegf,dfeh
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10 bcd,aefgh,ahig,agfj,bjgi,biidg,bfcedh,bgjfc,cfe,dhe
10 bcde,aefgh,ahif,ajhg,aijb,bjcg,bfid,bdjic,che,dfeh
10 bcd,aefg,aghi,afhj,bjih,bhdig,bfjhc,cgdf,cejf,dgie
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10 bcd,aefgh,ahfi,agfj,bjif,behdg,bfdih,bgjfc,ceg,dhe
10 bcd,aefgh,aij,afhge,bdgif,behdj,bjedh,bgdfi,che,cgf
11 bcd,aefg,ahf,aij,bji,bick,bkdh,cgij,dfeh,dkhe,fjg
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11 bcd,aefg,aghi,ajkg,bkh,bjig,bfkdhc,cgje,ckf,dfh,dgie
11 bcd,aefgh,aij,afhk,bkif,behdj,bjkh,bgdfi,che,cgf,dge
11 bcd,aefgh,aij,agfk,bkjf,beidg,bfdjh,bgki,chk,ceg,dhe
11 bcde,afgh,ahfi,aijk,akif,becj,bjik,bkjc,cegd,dgfh,dhge
12 bcd,aefg,ahf,aij,bki,bicl,blih,cgk,dfeg,dlk,ejh,fjg
12 bcd,aefg,ahi,ajkl,blj,bilk,bkjh,cgl,cfj,diag,dgf,dfhe
12 bcd,aefg,ahi,ajk,bkhl,bikg,bfjh,cgk,cfl,dlg,dfhe,eji
12 bcd,aefg,ahi,ajk,bkhf,beji,bikl,cle,ckgf,dfh,dgie,gjh
12 bcd,aefgh,aij,akl,blif,bekj,bjhl,bgki,che,cgf,dfh,dge
13 bcd,aefg,ahi,ajk,blm,bikg,bfjh,cgl,cfm,dmg,dfh,ekh,eji
14 bcd,aef,agh,aij,bkl,bmn,cnk,cml,dln,dmk,ejg,eih,fhj,fig
9 bcde,afdg,ageh,ahbi,aicf,begi,bhfc,cigd,dfhe
9 bcde,afdg,agfh,ahbi,aihfb,beci,bhic,cegd,dfge
13 bcde,afgh,ahij,ajkl,almf,beji,biml,blkc,ckgf,cfmd,dmih,dhge,egkj

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FIGURE 1. List of Diminimal Maps on the Torus in *plantri* format

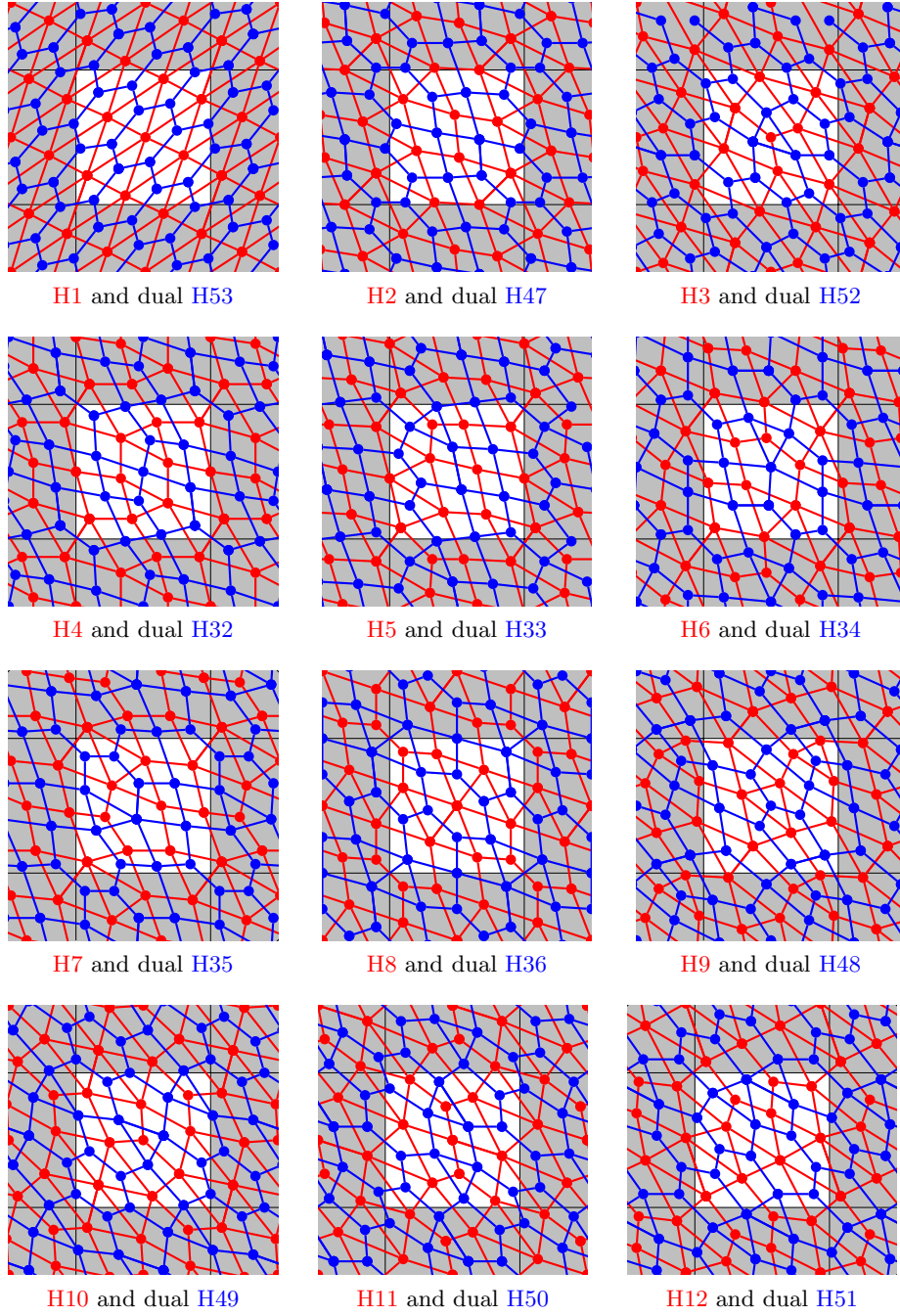


FIGURE 2. Diminimal Maps on the Torus with duals, 1 of 3

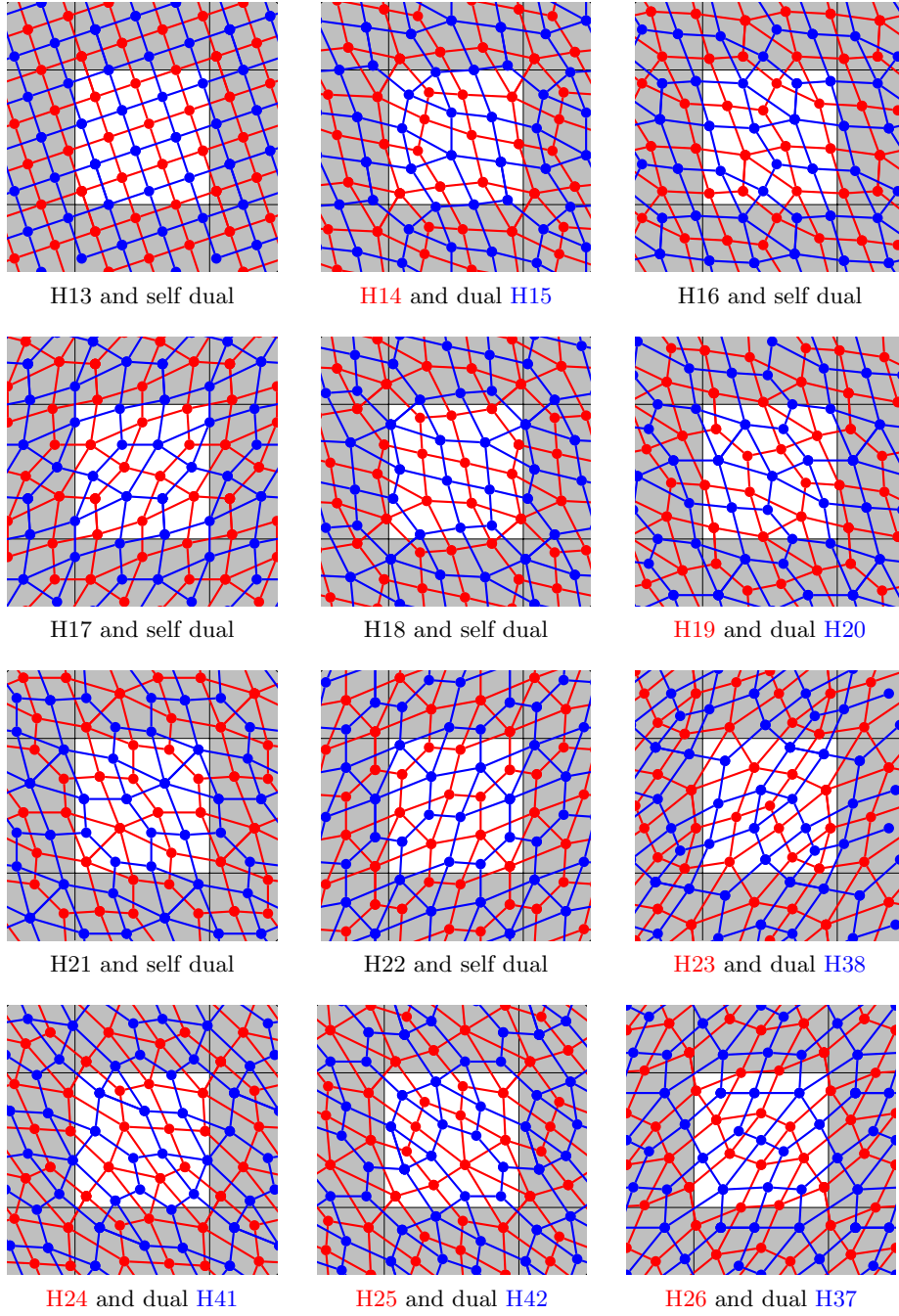


FIGURE 3. Diminimal Maps on the Torus with duals, 2 of 3

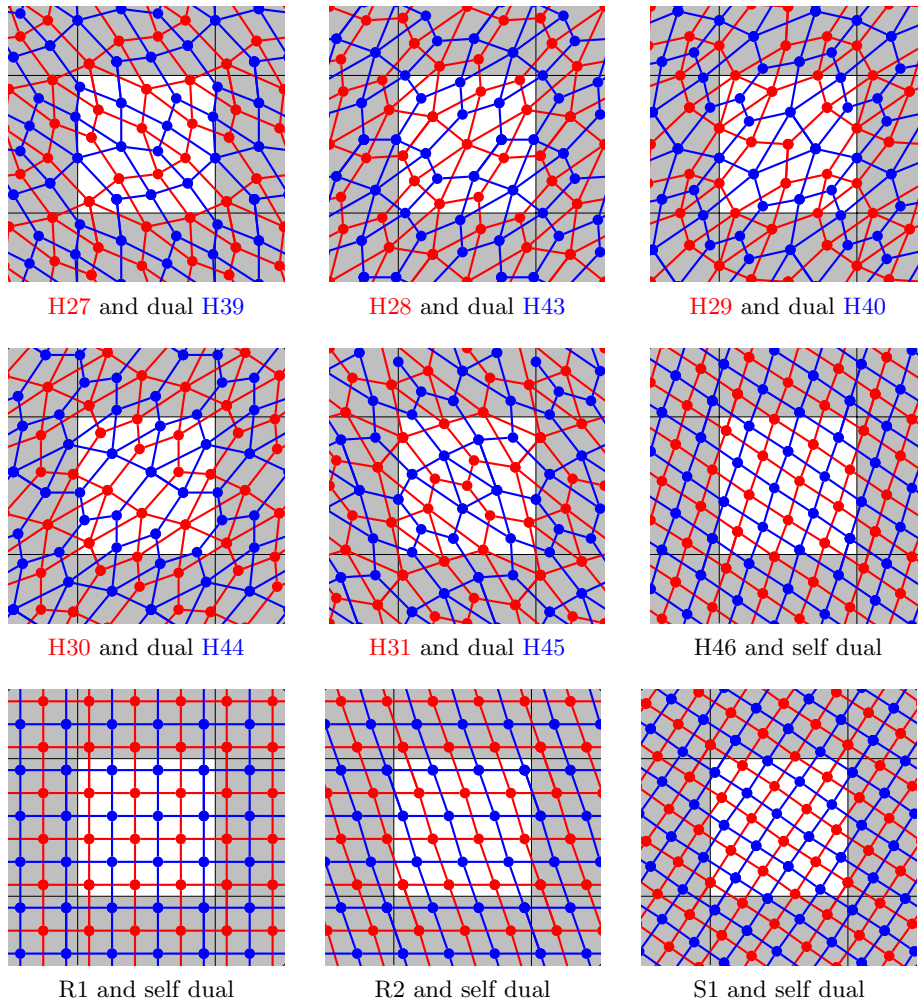


FIGURE 4. Diminimal Maps on the Torus with duals, 3 of 3

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